# CAPITAL BUDGETING WITH UNSPECIFIED DISCOUNT RATES*

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## Abstract

Public or private allocation problems often involve numerous multiperiod decision alternatives, conflicting preferences, or imperfect knowledge about decision-makers' wants. By placing only weak restrictions on permissible preferences, very simple cashflow characteristics are used in this paper to select most-preferred projects for any individual within the general class. Judging from an empirical test, the approach seems powerful in terms of simplifying complex capital budgeting problems. For instance, presupposing that discount rates are positive and constant over time, 26 out of 30 mutually exclusive projects could immediately be disregarded by every individual, regardless of what specific value is taken on by this discount rate.

## I. The Problem

Consider a set $X$ of mutually exclusive capital budgeting projects, where any project $x \in X$ is completely described by its cashflow $x$:[1]

$$x = \{x(t) \mid x(t) \in R, \quad 0 \leqslant t \leqslant T\}.$$

Here, $t=0$ denotes the present, $T$ is the horizon of the most long-lived project, $R$ is the set of real numbers, and any $x(t)$ may be deterministic or stochastic. With $a \in X$ and $b \in X$ denoting two projects with cashflows $a$ and $b$, respectively, the choice between them generally depends on the criterion used.

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[1] This paper deals only with mutually exclusive projects, as any choice between alternatives can be formulated in this way. Even if every project is independent of any other, the problem can be transformed into one of choosing between mutually exclusive projects by appropriately redefining the alternatives.

In the certainty case, consider an economy where consumer preferences and production technology satisfy the standard convexity assumptions. Debreu (1959) showed that if discount rates are interpreted as ratios between commodity prices at different points in time, and if the projects $a$ and $b$ do not affect equilibrium prices, a competitive equilibrium ensures that:

$$a \succ b \Leftrightarrow \text{NPV}\ (a, R_T) = \int_0^T a(t)\, e^{-tr(t)}\, dt > \text{NPV}\ (b, R_T) = \int_0^T b(t)\, e^{-tr(t)}\, dt. \qquad \text{(I)}$$

Here, NPV denotes net present value, $v(t) \equiv e^{-tr(t)}$ is the discounting function, $r(t)$ is the discount rate used for discounting from $t$ back to zero, and $R_T$ is the set of such rates from 0 to $T$.[1] Thus, every utility-maximizing individual ranks production opportunities as well as consumption paths according to their net present value, ultimately selecting the one with the highest NPV.

Dropping the assumption of a general, competitive equilibrium, a perfect capital market is nevertheless sufficient for separating consumer preferences from project evaluation; see Fisher (1930). Thus, once more, individual tastes are irrelevant, NPV can be used for ranking, and $r(t)$ equals the market rate. However, when lending rates differ from borrowing rates, Hirshleifer (1958) showed that lending rates should be used during periods when the investor is a lender, whereas borrowing rates apply when he prefers to be a borrower.

When no market assumptions are made, a valid use of NPV presupposes more than just convex preferences; cf. Koopmans (1960) and Williams & Nassar (1966). Several behavioral axioms must be satisfied if (I) is to ensure a utility-maximizing ranking, and $r(t)$ becomes a subjective concept, expressing a personal time preference.

When cashflows are stochastic, the general equilibrium results of the deterministic case still apply under perfect and complete markets for state-contingent claims, although the NPV is now found by integrating over states as well as time; see Hirshleifer (1970). Recently, however, Fama (1977) showed that in a capital market where assets are valued in each period according to the capital asset pricing models of Sharpe (1964), Lintner (1965) and Black (1972), the market value of a multiperiod, stochastic cashflow may be expressed as the NPV of the expected cashflow, using risk-adjusted discount rates. Moreover, in *ad hoc* uncertainty models based on NPV, similar risk adjustments are made through the determination of $R_T$; see Bierman & Smidt (1975) and Lewellen (1977). Thus, even under uncertainty, (I) is still valid for project selection, if $a(t)$ and $b(t)$ are interpreted as expected values and it is

[1] Letting $\varrho(s)$ denote the instantaneous rate from $s$ to $s+ds$, $v(t)$ may be stated alternatively as

$$v(t) = e^{-tr(t)} = \exp\left\{-\int_0^t \varrho(s)\, ds\right\}.$$

Thus, $r(t) = 1/t \int_0^t \varrho(s)\, ds$, which may be interpreted as an "average" of the instantaneous rates from 0 to $t$.

kept in mind that the discounting function involves adjustments for time as well as risk.

Now, in the market contexts of Debreu, Fisher or Fama, the NPV criterion yields a correct ranking for any investor if the appropriate market parameters are used for discounting in (I). In the three remaining cases, however, a straightforward use of NPV is more problematic.

First, in an imperfect capital market, the appropriate discount rates cannot be observed until the preferred allocation has been chosen, and the resulting ranking is only valid for individuals who follow exactly this borrowing/lending path. Second, when the validity of NPV is based on behavioral axioms only, the discount rates are subjective and cannot be observed in the market. Consequently, ranking according to a particular $R_T$ applies only to individuals with that specific preference structure. Third, in *ad hoc* uncertainty models, $R_T$ is both project and investor-specific, as the discount rates reflect project risk as well as risk attitude. Therefore, in all three cases, the numerical values of $r(t) \in R_T$ must be completely specified before (I) is used. This in itself may offer a difficult estimation problem. Furthermore, after $R_T$ has been established, the ranking tends to be relevant for only a very limited set of individuals.

Due to these problems of data availability and ranking generality, an alternative approach will be suggested which does not require a prespecified set of discount rates. Nevertheless, it is consistent with NPV, it reduces that method's inherent estimation problem and increases the generality of the ultimate ranking.[1] For expositional convenience, we consider only deterministic projects and the preference axiomatic basis for NPV. By appropriately reinterpreting $x(t)$ and $R_T$, every conclusion is valid for any of the remaining cases.[2]

Next, three classes of preference orderings are discussed, after which so-called time dominance properties of a cashflow are defined in Section III. The major results are presented in Sections IV and V, where NPV-consistent ranking rules are derived by relating classes of preference orderings to time dominance properties. After an empirical test in Section VI, the findings are summarized in the final section.

## II. The Discounting Function

Consider the discounting function $v(t)$, which is assumed to have continuous derivatives of first and second order. Three different classes of such functions are defined in the following.

[1] Under uncertainty, mean/variance, mean/semivariance and stochastic dominance criteria are used to resolve such issues; cf. Markowitz (1959), Porter (1974) and Bawa (1975). However, either approach delimits its attention to risky decision problems with a one-period horizon.

[2] Under uncertainty, it must be assumed that every project belongs to the same risk class, ensuring that when $a$ and $b$ are compared, the expected cashflows are discounted by the same set of risk-adjusted discount rates; cf. Fama (1977).

*Class 1.* Discounting functions that are decreasing functions of $t$. This class consists of any $v(t)$ where

$$\dot{v}(t) = \frac{\partial v}{\partial t} < 0, \quad 0 \leqslant t \leqslant T.$$

Class 1 contains all discounting functions that exhibit positive time preference, i.e. a dollar at time $s$ is preferred to a dollar at time $s+\Delta(\Delta>0)$. The set of functions belonging to this class is denoted $V^1$.

*Class 2.* Discounting functions that are decreasing and convex functions of $t$. In this class, denoted $V^2$, it is required of every $v(t)$ that

$$\dot{v}(t) < 0, \ddot{v}(t) = \frac{\partial^2 v}{\partial t^2} \geqslant 0, \quad 0 \leqslant t \leqslant T.$$

Thus, for any discounting function to be a member of the second class, it must exhibit positive time preference (as in $V^1$) and decrease with $t$ in a convex fashion.

*Class 3.* Discounting functions with a positive, constant discount rate. Class $V^3$ contains functions of the type

$$v(t) \equiv e^{-rt} \quad (r>0).$$[1]

Reviewing the three classes, the relationship between them is seen to be

$$V^3 \subseteq V^2 \subseteq V^1. \tag{II}$$

Here, $V^1$ contains the largest and $V^3$ the smallest set of functions, with $V^2$ somewhere in between. In other words, $V^1$ makes the weakest and $V^3$ the strongest assumptions about the shape of admissible discounting functions.

Having classified investor preferences by means of these three classes, we now turn to the opportunity set by defining some cashflow characteristics.

## III. Time Dominance

Consider two cashflows $a \in X$ and $b \in X$, as defined previously. Then, $a$ is said to dominate $b(a\mathrm{D}b)$ by *first-order time dominance* (1TD) if

$$F^1(s) = G_a^1(s) - G_b^1(s) = \int_0^s a(t)\,dt - \int_0^s b(t)\,dt \geqslant 0 \quad \text{for any } s,$$

$0 \leqslant s \leqslant T$, and with strict inequality for some $s$. (IIIa)

[1] It follows from the definition of $r(t)$ in footnote 1, p. 46, that $v(t) = -\dot{v}(t)\varrho(t)$ and $\ddot{v}(t) = -v(t)\dot{\varrho}(t) - \dot{v}(t)\varrho(t)$. As $v(t)>0$, any discounting function in $V^1$ has positive instantaneous rates for every $t$. Moreover, when $\dot{v}(t)<0$, the condition $\dot{\varrho}(t) \leqslant 0$ is sufficient, but not necessary, to ensure $\ddot{v}(t) \geqslant 0$ for every $t$. Thus, $V^2$ is larger than the class of functions with positive, nonincreasing rates of marginal time preference. Finally, inserting the constant $\varrho(s) = \bar{\varrho}$ in the definition of $r(t)$ demonstrates that $V^3$ contains any discounting function with positive, constant instantaneous rates.

Moreover, $a$ is said to dominate $b(aDb)$ by *second-order time dominance* (2TD) if

(1) $F^1(T) \geqslant 0$, and

(2) $F^2(s) = G_a^2(s) - G_b^2(s) = \int_0^s G_a^1(t)\,dt - \int_0^s G_b^1(t)\,dt \geqslant 0$ for any $s$,

$0 \leqslant s \leqslant T$, and either (1) holds as a strict inequality or (2) holds as a strict inequality for some $s$. (IIIb)

Generally, $a$ is said to dominate $b(aDb)$ by *n'th order time dominance* ($n$TD) if

(1) $F^1(T) \geqslant 0$,

(2) $F^2(T) \geqslant 0$,

⋮

$(n-1)$ $F^{n-1}(T) \geqslant 0$, and

$(n)$ $F^n(s) = \int_0^s F^{n-1}(t)\,dt \geqslant 0$ for any $s$, $0 \leqslant s \leqslant T$,

and either some of (1) to $(n-1)$ holds as a strict inequality or $(n)$ holds as a strict inequality for some $s$. (IIIc)

Definition (IIIc) applies to every integer $n$ larger than or equal to unity, defining $F^0(s) \equiv G_a^0(s) - G_b^0(s) = a(s) - b(s)$. Thus, (IIIa) and (IIIb) are special cases of (IIIc), with $n=1$ and $n=2$, respectively.

Verbally stated, $F^1(s)$ is the cumulative cashflow of $a$ less the cumulative cashflow of $b$ from time 0 up to time $s$. Correspondingly, $F^2(s)$ denotes the cumulative of $F^1(s)$, $F^3(s)$ is the cumulative of $F^2(s)$, etc. Moreover, it is easily seen that 1TD implies 2TD, but not *vice versa*. Consequently, if $a$ dominates $b$ by 1TD, it will always dominate $b$ by 2TD. However, $aDb$ by 2TD does not imply that $aDb$ by 1TD. Generally,

$aDb$ by $n$TD $\Rightarrow aDb$ by $k$TD, $n = 1, 2, 3, \ldots$; integer $k \geqslant n$. (IVa)

Letting D{$n$TD} $\in X$ denote the set of dominating projects according to criterion $n$TD, D{3TD} will therefore be a subset of D{2TD}, which in turn is contained in D{1TD}.[1] Generally,

D{$k$TD} $\subseteq$ D{$n$TD} $n = 1, 2, 3, \ldots$; integer $k \geqslant n$. (IVb)

Thus, 2TD is a stronger criterion than 1TD, as more cases can be ordered according to the former.

## IV. Most-Preferred Projects and Time Dominance

In this section, the three classes of discounting functions are related to the time dominance properties of cashflows, thereby yielding ranking rules which

[1] When D{$n$TD} $= X$, the term "dominating projects" may seem semantically inappropriate, as there is no project being dominated. However, it will be seen later that if such a case occurs, the value of the proposed procedure is zero.

are consistent with NPV, but which do not require a full specification of discount rates. The proofs of the following propositions are provided in Appendix 1.

**Proposition 1.** *Let the discounting function $v(t)$ be a member of $V^1$. Then every individual prefers a to b if a dominates b by first-order time dominance.*

Thus, if a dollar today is preferred to one tomorrow, $a$ will always be better than $b$ if $aDb$ by 1TD, i.e. if the cumulative cashflow of $a$ is at least as large as that of $b$, and strictly larger at least once. Given the set $X$ of mutually exclusive projects, the most-preferred investment for every $v(t) \in V^1$ will always be a member of D{1TD}, and those $x \notin$ D{1TD} need not be considered by anyone.

The advantage of Proposition 1 is its nonrestrictive assumptions about admissible discounting functions. The obvious weakness is that the set D{1TD} may be large relative to $X$, so that the decision problem is simplified to only a very limited extent.

**Proposition 2.** *Let the discounting function $v(t)$ be a member of $V^2$. Then every investor prefers a to b if a dominates b by second order time dominance.*

Compared to Proposition 1, a stronger ranking criterion is obtained at the expense of being relevant for a smaller class of investors. The same relationship holds between Propositions 2 and 3.

**Proposition 3.** *Let the discounting function $v(t)$ be a member of $V^3$. Then every investor prefers a to b if a dominates b by n'th order time dominance, $n=3$ or 4 or 5 or ....*

Of course, Proposition 3 is also valid for $n=1$ or 2, as both 1TD and 2TD imply $n$TD for $n \geq 3$. Having restricted admissible preferences to $V^3$, however, as strong a ranking criterion as possible should be used. For the same reason, 1TD is not used in Proposition 2.

There is a remarkable thing about Proposition 3 which has no parallel in the theory of stochastic dominance. Going from Proposition 1 through 2 to 3, more powerful criteria are obtained by successively delimiting the class of admissible discounting functions. Having reached $V^3$, however, the power of the criterion can be increased freely (in TD terms) without having to place stronger restrictions on preferences. Therefore, a dominating set of any order will always contain the most-preferred project for any discounting function with constant discount rates. It should be noted that the assumption of constant discount rates is very frequent in the theory of capital budgeting and finance; cf. Bierman and Smidt (1975), Hirshleifer (1970) and Fama and Miller (1972). In fact, it is often the only case considered.

Next, some simple examples are given, using discrete cashflows only.

*Example 1.* Let $a=(-1, -1, 3, 2, 3)$ and $b=(-1, -2, 3, 3, 2)$, where the first vector element is at $t=0$ and the last at $t=4$. Then, $G_a^1=(-1, -2, 1, 3, 6)$ and $G_b^1=(-1, -3, 0, 3, 5)$, giving $F^1=(0, 1, 1, 0, 1)$. Thus, $a$D$b$ by 1TD, and every individual with positive time preference prefers $a$ to $b$.

*Example 2.* Let $a$ be as in example 1, and $b=(-2, 0, 4, 1, 2)$. Then, $G_b^1=(-2, -2, 2, 3, 5)$, $F^1=(1, 0, -1, 0, 1)$, and there is no 1TD. However, $F^1(4)>0$ and $F^2=(1, 1, 0, 0, 1)$, and $a$D$b$ by 2TD. Consequently, every investor whose discounting function is a decreasing and convex function of $t$ will prefer $a$ to $b$.

*Example 3.* Let $a$ still be as in example 1, and $b=(-2, 0, 4.1, 1, 2)$. Then $G_b^1=(-2, -2, 2.1, 3.1, 5.1)$, $F^1=(1, 0, -1.1, -0.1, 0.9)$ and $F^2=(1, 1, -0.1, -0.2, 0.7)$, so there is neither first nor second-order time dominance. However, $F^1(4)>0$, $F^2(4)>0$ and $F^3=(1, 2, 1.9, 1.7, 2.4)$ contains only positive elements, so that $a$D$b$ by 3TD. This means that for every discount rate $r$, $0<r<\infty$, $\mathrm{NPV}(a, r)>\mathrm{NPV}(b, r)$. Alternatively stated, the NPV of the differential flow $(a-b)$ is positive for every discount rate greater than zero, and the flow $(a-b)$ has no positive internal rate of return.

## V. Most-Preferred Projects and Normalized Time Dominance

The TD criteria have a serious drawback when it comes to their ability to minimize the size of the dominating set. As can be seen from their definitions as well as from the numerical examples, a large project cannot dominate a small one, when project size is measured in terms of initial investment. In other words, when $a(0)<b(0)$, $a$ can never dominate $b$ by any TD criterion, regardless of the remaining cashflow elements of $a$ and $b$. For instance, if $a=(-1, 1\,000, 1\,000)$ and $b=(-0.9, 0, 1)$, the projects cannot be ranked. Thus, if no other project dominates them, they will both appear in the dominating set. This section introduces normalized time dominance (NTD) criteria for the purpose of ranking such projects.

The undiscounted value of project $x\in X$ is $G_x^1(T)$, i.e. the total cashflow generated during its life. It will be assumed that $G_x^1(T)>0$ for every $x\in X$, i.e. that the total cash inflows exceed the total outflows. The *normalized cashflow* $\hat{x}$ is then defined as

$$\hat{x}\equiv\{\hat{x}(t)=x(t)/G_x^1(T)\,|\,\hat{x}(t)\in R,\quad 0\leq t\leq T\}.$$

For instance, in the discrete case, if $x=(-1, 3, 8)$, then $G_x^1(T)=10$ and $\hat{x}=(-0.1, 0.3, 0.8)$. Furthermore, the following definitions of cumulative, normalized flows are needed:

$\hat{G}_a^0(s) \equiv \hat{a}(s), \quad \hat{G}_b^0(s) \equiv \hat{b}(s),$
$\hat{F}^0(s) \equiv \hat{G}_a^0(s) - \hat{G}_b^0(s),$

and for any integer $n \geq 1$,

$$\hat{G}_a^n(s) \equiv \int_0^s \hat{G}_a^{n-1}(t)\,dt, \quad \hat{G}_b^n(s) \equiv \int_0^s \hat{G}_b^{n-1}(t)\,dt, \quad \hat{F}^n(s) \equiv \int_0^s \hat{F}^{n-1}(t)\,dt\, \hat{G}_a^n(s) - \hat{G}_b^n(s).$$

From these definitions, the counterpart of (IIIc) can be established: $a$ is said to dominate $b$ ($a\mathrm{D}b$) by $n$'th order normalized time dominance ($n$ NTD) if

(1) $\hat{F}^1(T) \geq 0,$
(2) $\hat{F}^2(T) \geq 0,$
$\vdots$
$(n-1)$ $\hat{F}^{n-1}(T) \geq 0,$ and
$(n)$ $\hat{F}^n(s) = \int_0^s \hat{F}^{n-1}(t)\,dt \geq 0,$ for any $s$, $0 \leq s \leq T$,

and either some of (1) to $(n-1)$ holds as a strict inequality or $(n)$ holds as a strict inequality for some $s$. (V)

Here, (1) is redundant, as $\hat{G}_a^1(T) = \hat{G}_b^1(T) = 1$ by definition. Letting $\mathrm{D}\{n\mathrm{NTD}\} \in X$ denote the set of dominating projects according to criterion $n$NTD, it follows that

$$\mathrm{D}\{k\mathrm{NTD}\} \subseteq D\{n\mathrm{NTD}\} \quad n = 1, 2, 3, \ldots; \text{ integer } k \geq n. \tag{VI}$$

Thus, as was found in the case of nonnormalized time dominance (TD), the higher the order of the NTD, the stronger it is. The proofs of the following propositions are given in Appendix 1.

**Proposition 4.** *Let the discounting function $v(t)$ be a member of $V^1$. Suppose that $G_a^1(T) \geq G_b^1(T) > 0$ and that $NPV(b, R_T) > 0$. Then every investor prefers a to b if a dominates b by first-order normalized time dominance.*

**Proposition 5.** *Let the discounting function $v(t)$ be a member of $V^2$. Suppose that $G_a^1(T) \geq G_b^1(T) > 0$ and that $NPV(b, R_T) > 0$. Then every investor prefers a to b if a dominates b by second-order normalized time dominance.*

**Proposition 6.** *Let the discounting function $v(t)$ be a member of $V^3$. Suppose that $G_a^1(T) \geq G_b^1(T) > 0$ and that $NPV(b, R_T) > 0$. Then every investor prefers a to b if a dominates b by n'th order normalized time dominance, $n \equiv 3$ or 4 or 5 or ...*

Each of the above propositions presupposes that $\mathrm{NPV}(b, R_T) > 0$. This seems to be a very strange assumption, as the whole idea behind the criteria is to establish rules that do not require fully specified discount rates. However, it will generally hold that if $G_a^1(T) \geq G_b^1(T) > 0$ and $a\text{D}b$ by $n$NTD, $n = 1, 2, 3, \ldots$, then either $\text{NPV}(b, R_T) \leq 0$ or $0 < \text{NPV}(b, R_T) < \text{NPV}(a, R_T)$ for every $v(t)$ within the appropriate class. In either case, project $b$ should be rejected. Consequently, the normalized time dominance criteria can be used without *a priori* knowledge of the numerical values of discount rates.

Because of Propositions 4, 5 and 6, there are now two dominance criteria for each set of discounting functions, a nonnormalized criterion and a normalized one. For instance, if $v(t) \in V^2$ and the total cashflow of $a$ is at least as large as the positive total cashflow of $b$, the dominating sets D{2TD} and D{2NTD} will both contain the most-preferred project for any $v(t) \in V^2$, provided that there is at least one project in $X$ that has a nonnegative NPV.[1] It is easily seen that if $a\text{D}b$ by $n$TD, it does not follow that $a\text{D}b$ by $n$NTD, and *vice versa*. Consequently, as the two criteria provide different sufficient conditions for $a$ to be preferred to $b$, the most-preferred project must be contained in the *intersection* of the two dominating sets, $D^*\{n\}$, defined as

$$\text{D}^*\{n\} = \text{D}\{n\text{TD}\} \cap \text{D}\{n\text{NTD}\}. \tag{VII}$$

Thus, by combining the two criteria and considering only projects that appear in both dominating sets, the size of the ultimate dominating set is minimized. This point is illustrated in the empirical study presented in the next section.

## VI. An Empirical Test

In order to illustrate the criteria and evaluate their ranking power, 30 capital budgeting projects taken from Weingartner (1963) are presented in Appendix 2. As they originally have differing implementation dates, the cashflows of the projects starting after $t = 0$ have been moved backwards in time, ensuring that $t = 0$ is the common starting date. Moreover, some of the cashflows have been cut, allowing for a 10-year planning horizon. Nevertheless, the 30 projects still differ in scale, cashflow and internal rate of return, and it is far from obvious what the best project should be for a given class of individuals.

The dominating sets are reported in Table 1.[2] It is seen from the *first column*

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[1] Because the "do nothing" alternative with only zero cashflow elements is assumed to be included in $X$, this condition is automatically satisfied, as the NPV of that project is zero.

[2] As the TD and NTD criteria are very easily programmed, the tests were performed by computer. In the majority of practical cases, however, $X$ contains considerably less than 30 mutually exclusive projects. Then, the simple cashflow accumulations are more efficiently performed by hand or calculator.

Notice the simplicity of this procedure versus one of calculating NPVs of every project with every admissible set of discount rates. In principle, of course, the latter approach is infeasible, as the set of discount rates is infinite. Even when computerizing the test and using a finite subset of rates, such a procedure seems clearly inferior in cost benefit terms.

Table 1. *Dominating sets from 30 capital budgeting projects, based on various time dominance criteria*

| Order(n) | Dominance criterion | | |
|---|---|---|---|
| | Non-normalized time dominance, giving D{nTD} | Normalized time dominance, giving D{nNTD} | Combined criterion, giving D*{n} |
| 1 | {1, 5, 7, 8, 9, 14, 16, 19, 23, 24} | {5, 7, 8, 9, 14, 22, 23, 24, 28} | {5, 7, 8, 9, 14, 23, 24} |
| 2 | {1, 5, 7, 8, 9, 14, 16, 19, 23, 24} | {5, 9, 14, 23} | {5, 9, 14, 23} |
| ⩾3 | {1, 5, 7, 8, 9, 14, 16, 19, 23, 24} | {5, 9, 14, 23} | {5, 9, 14, 23} |

of nonnormalized TD that under an assumption of positive time preference, the decision-maker may consider only 10 of the 30 projects. Moreover, this dominating set cannot be reduced by placing more restrictions on $v(t)$ and applying a stronger ranking criterion. For instance, if 10TD is used, the dominating set is still identical to D{1TD}. Referring to expression (IVb), the present example is the limiting case of D{nTD} = D{1TD} for any integer $n \geqslant 1$.

Turning to the results of normalized time dominance in the *second column*, the number of projects relevant for each investor's ultimate choice drops from 9 to 4 when going from 1NTD to 2NTD, and remains constant thereafter. Thus, in this case, NTD yields a smaller dominating set than TD, particularly when $v(t)$ belongs to $V^2$ or $V^3$ (4 versus 10 projects).

The dominating set of the combined criterion, D*{n}, appears in the *third column*. First, notice that D{1TD} is not a subset of D{1NTD} or *vice versa*. Second, going from 1TD through 1NTD to the first-order combined criterion, the number of elements in the dominating set drops from 10 through 9 to 7. Moreover, as D*{n} always contains the most-preferred project for any admissible $v(t)$, anyone may delimit his attention to 7 projects if his $v(t) \in V^1$, and to 4 projects if $v(t)$ belongs to $V^2$ or $V^3$. Compared to the initial set of 30 projects, the time dominance tests have simplified the problem to a remarkable extent.

In sum, merely by examining simple cashflow properties, the decision-maker does not have to specify his preferences relative to $X$, but only relative to the generally smaller D*{n}. Consequently, the benefit of this approach is greater, the fewer the elements in D*{n}.

If D*{n} contains only one project, the problem is solved. Otherwise, a local specification of $v(t)$ is called for, either explicitly or implicitly.[1] In that case, which is probably the most common one, a word of caution may be in order.

[1] An implicit specification would be e.g. a case where the final choice is made on the basis of financial policy restrictions.

If, due to a misspecified $v(t)$, a project is selected from $D^*\{n\}$ which is not the best one (according to the true $v(t)$), this alternative may be less preferred than several projects *not* included in $D^*\{n\}$. In other words, for any specific $v(t)$, the corresponding $D^*\{n\}$, containing say $m$ projects, is generally not identical to the $m$ projects in $X$ with the largest NPV (although, of course, the project with maximum NPV is always in $D^*\{n\}$).

## VII. Summary

This paper has established time dominance criteria for choice between deterministic or stochastic capital budgeting projects. Three sets of general assumptions were made about discounting functions, the strongest one being that a positive, constant discount rate is used. As to project characteristics, nonnormalized (TD) as well as normalized time dominance (NTD) criteria were constructed, utilizing cashflow properties that are very easily derived.

Relating the TD and NTD criteria to assumptions about discounting functions, we found sufficient conditions for any decision-maker to prefer one project to another, allowing for the construction of dominating sets. Due to a particular relationship between TD and NTD, the intersection of their respective dominating sets will generally contain the smallest number of projects.

Finally, the various criteria were tested on 30 deterministic projects. Presupposing only that the decision-maker prefers a dollar today to a dollar tomorrow (whatever the specific set of positive discount rates), more than two thirds of the projects could be eliminated. Under two more restrictive sets of investor preference assumptions (the strongest being a positive, constant discount rate), only four out of 30 projects had to be evaluated by any investor.

Judging from this test, the criteria seem powerful in terms of simplifying complex capital budgeting problems. To the extent that the findings have general validity, the approach may effectively reduce the long-debated problem of discount rate specification.

## Appendix 1. Proofs of Propositions 1–6

*Proposition 1*

From (I) it follows that $a$ is preferred to $b$ if and only if:

$$H=\int_0^T a(t)\,v(t)\,dt-\int_0^T b(t)\,v(t)\,dt=\int_0^T (a(t)-b(t))\,v(t)\,dt>0.$$

Integrating by parts,

$$H=F^1(t)\,v(t)\Big|_0^T-\int_0^T F^1(t)\,\dot{v}(t)\,dt=F^1(T)\,v(T)-\int_0^T F^1(t)\,\dot{v}(t)\,dt.$$

The discounting function $v(t)$ is positive for every $t$, and $\dot{v}(t)<0$ by assumption. Consequently, $H>0$ if $a\mathrm{D}b$ by 1TD as defined in (IIIa) or (IIIc).

*Proposition 2*

Using the terminology from the proof of Proposition 1, two integrations by parts yield:

$$H=F^1(T)\,v(T)-F^2(T)\,\dot{v}(T)+\int_0^T F^2(t)\,\ddot{v}(t)\,dt.$$

By assumption, $v(T)>0$, $\dot{v}(T)<0$, $\ddot{v}(t)\geqslant 0$. Consequently, $H$ is positive if $a\mathrm{D}b$ by 2TD as defined in (IIIb) or (IIIc).

*Proposition 3*

$$H=e^{-Tr}[F^1(T)+rF^2(T)+\ldots+r^{n-1}F^n(T)]+r^n\int_0^T F^n(t)\,e^{-tr}\,dt.$$

As $r$ is positive by assumption, $H$ is positive if $a\mathrm{D}b$ by $n$TD, $n=3$ or 4 or 5 or ..., as defined in (IIIc).

*Proposition 4*

According to (I), a necessary and sufficient condition is

$$\int_0^T a(t)\,v(t)\,dt>\int_0^T b(t)\,v(t)\,dt.$$

Multiplying by $G_a^1(T)/G_a^1(T)$ on the left-hand and by $G_b^1(T)/G_b^1(T)$ on the right-hand side, the condition is

$$G_a^1(T)\int_0^T \hat{a}(t)\,v(t)\,dt>G_b^1(T)\int_0^T \hat{b}(t)\,v(t)\,dt.$$

Multiplying through by the positive $1/G_b^1(T)$

$$\frac{G_a^1(T)}{G_b^1(T)}\int_0^T \hat{a}(t)\,v(t)\,dt>\int_0^T \hat{b}(t)\,v(t)\,dt.$$

After one integration by parts, the condition is

$$\frac{G_a^1(T)}{G_b^1(T)}\left[v(T)-\int_0^T \hat{G}_a^1(t)\,\dot{v}(t)\,dt\right]>\left[v(T)-\int_0^T \hat{G}_b^1(t)\,\dot{v}(t)\,dt\right].$$

By assumption, the right-hand side is positive ($\mathrm{NPV}(b, R_T)>0$), $G_a^1(T)\geqslant G_b^1(T)$, and $\dot{v}(t)<0$ for all $t$. Then, the condition holds if $a\mathrm{D}b$ by 1NTD as defined by ($V$).

*Proposition 5*

Integrating by parts once more on the condition from the preceding proof,

$$\frac{G_a^1(T)}{G_b^1(T)}\left[v(T)-\hat{G}_a^2(T)\,\dot{v}(T)+\int_0^T \hat{G}_a^2(t)\,\ddot{v}(t)\,dt\right]$$

$$>\left[v(T)-\hat{G}_b^2(T)\,\dot{v}(T)+\int_0^T \hat{G}_b^2(t)\,\ddot{v}(t)\,dt\right].$$

By assumption, the right-hand side is positive, $G_a^1(T)\geqslant G_b^1(T)$, $\dot{v}(t)<0$ and $\ddot{v}(t)\geqslant 0$ for all $t$. Then, the inequality holds if $a\mathrm{D}b$ by 2NTD as defined by (V).

*Proposition 6*

After $n$ integrations on (I) with $v(t)=e^{-rt}$, the condition is

$$\frac{G_a^1(T)}{G_b^1(T)}\left\{e^{-rT}[1+r\hat{G}_a^2(T)+r^2\hat{G}_a^3(T)+\ldots+r^{n-1}\hat{G}_a^n(T)]+r^n\int_0^T \hat{G}_a^n(t)\,e^{-rt}\,dt\right\}$$

$$>e^{-rT}[1+r\hat{G}_b^2(T)+r^2\hat{G}_b^3(T)+\ldots+r^{n-1}\hat{G}_b^n(T)]+r^n\int_0^T \hat{G}_b^n(t)\,e^{-rt}\,dt.$$

By assumption, the right-hand side is positive, $G_a^1(T)\geqslant G_b^1(T)$, and $r>0$. Therefore, the proposition holds if $a\mathrm{D}b$ by $n$NTD as defined by ($V$).

## Appendix 2. 30 capital budgeting projects, taken from Weingartner (1963, p. 180), somewhat adjusted.

| Project number | Flow in year 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Positive internal rate of return |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | −100 | 20 | 20 | 20 | 19 | 19 | 18 | 16 | 14 | 11 | 6 | .12 |
| 2 | −100 | 20 | 18 | 18 | 18 | 18 | 14 | 14 | 14 | 14 | 14 | .11 |
| 3 | −100 | 15 | 15 | 15 | 15 | 15 | 13 | 13 | 13 | 13 | 13 | .07 |
| 4 | −100 | 20 | 6 | 11 | 7 | 16 | 5 | 14 | 18 | 3 | 20 | .03 |
| 5 | −100 | − 60 | − 60 | 80 | 74 | 66 | 56 | 44 | 30 | 14 | 0 | .13 |
| 6 | −200 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | .04 |
| 7 | − 80 | 20 | 20 | 20 | 19 | 17 | 14 | 10 | 6 | 2 | 0 | .14 |
| 8 | − 60 | − 30 | − 10 | 45 | 34 | 25 | 16 | 12 | 8 | −20 | 21 | .09 |
| 9 | −120 | 25 | 25 | 30 | 35 | 30 | 25 | 20 | 15 | 10 | 5 | .16 |
| 10 | −100 | 18 | 17 | 15 | 12 | 8 | −10 | 18 | 17 | 15 | 12 | .04 |
| 11 | −150 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | .06 |
| 12 | −100 | 20 | 18 | 16 | 14 | 12 | 10 | 4 | −20 | 20 | 18 | .03 |
| 13 | −150 | − 75 | − 75 | 60 | 60 | 55 | 50 | 44 | 38 | 36 | 35 | .05 |
| 14 | − 50 | −100 | −175 | 50 | 55 | 60 | 65 | 60 | 50 | 40 | 30 | .05 |
| 15 | −100 | −150 | −100 | 10 | 20 | 30 | 40 | 60 | 60 | 60 | 60 | None |

Appendix 2. (*Cont.*)

| Project number | Flow in year 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Positive internal rate of return |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | −95 | −60 | 47 | 42 | 37 | 31 | 24 | 18 | 13 | 9 | 6 | .10 |
| 17 | −175 | 50 | 45 | 35 | 25 | 10 | −60 | 45 | 35 | 25 | 10 | .06 |
| 18 | −250 | 45 | 45 | 40 | 30 | 25 | 20 | 15 | 10 | −40 | 40 | None |
| 19 | −75 | −75 | −40 | 40 | 40 | 40 | 35 | 35 | 30 | 25 | 15 | .06 |
| 20 | −180 | 20 | 12 | 16 | 13 | 11 | 19 | 17 | 12 | 15 | 19 | None |
| 21 | −80 | 18 | 16 | 14 | 12 | 10 | 4 | 16 | 14 | 10 | 6 | .09 |
| 22 | −85 | 20 | 20 | 16 | 15 | 13 | 10 | 7 | 3 | 0 | 0 | .06 |
| 23 | −270 | −100 | 125 | 115 | 105 | 80 | 60 | 35 | 25 | 15 | 10 | .12 |
| 24 | −40 | 15 | 13 | 9 | 7 | 5 | 2 | 0 | 0 | 0 | 0 | .10 |
| 25 | −50 | 10 | 10 | 9 | 7 | 4 | −14 | 9 | 9 | 8 | 6 | .03 |
| 26 | −200 | 60 | 40 | 30 | 15 | −25 | −25 | 50 | 40 | 30 | 20 | .03 |
| 27 | −70 | 15 | 13 | 11 | 10 | 9 | 7 | 6 | 4 | 3 | 2 | .03 |
| 28 | −335 | 60 | 70 | 80 | 70 | 55 | 40 | 25 | 15 | 5 | 0 | .06 |
| 29 | −275 | 40 | 45 | 45 | 40 | 35 | 30 | 25 | 20 | 15 | −75 | None |
| 30 | −140 | 20 | 20 | 18 | 16 | 14 | 11 | 8 | −25 | 18 | 18 | None |

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